
CS380: Computer Graphics 3D Transformation

Sung-Eui Yoon
(윤성익)

Course URL:
<http://sglab.kaist.ac.kr/~sungeui/CG>

KAIST

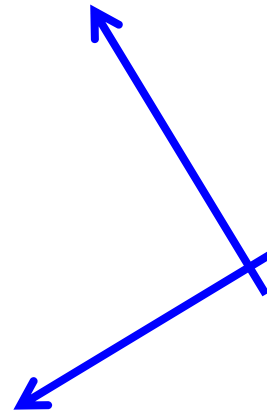
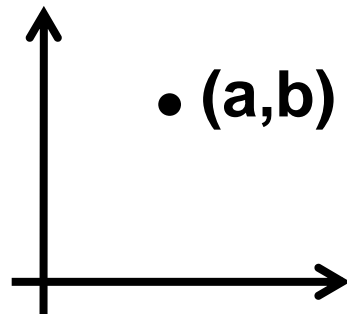


Class Objectives

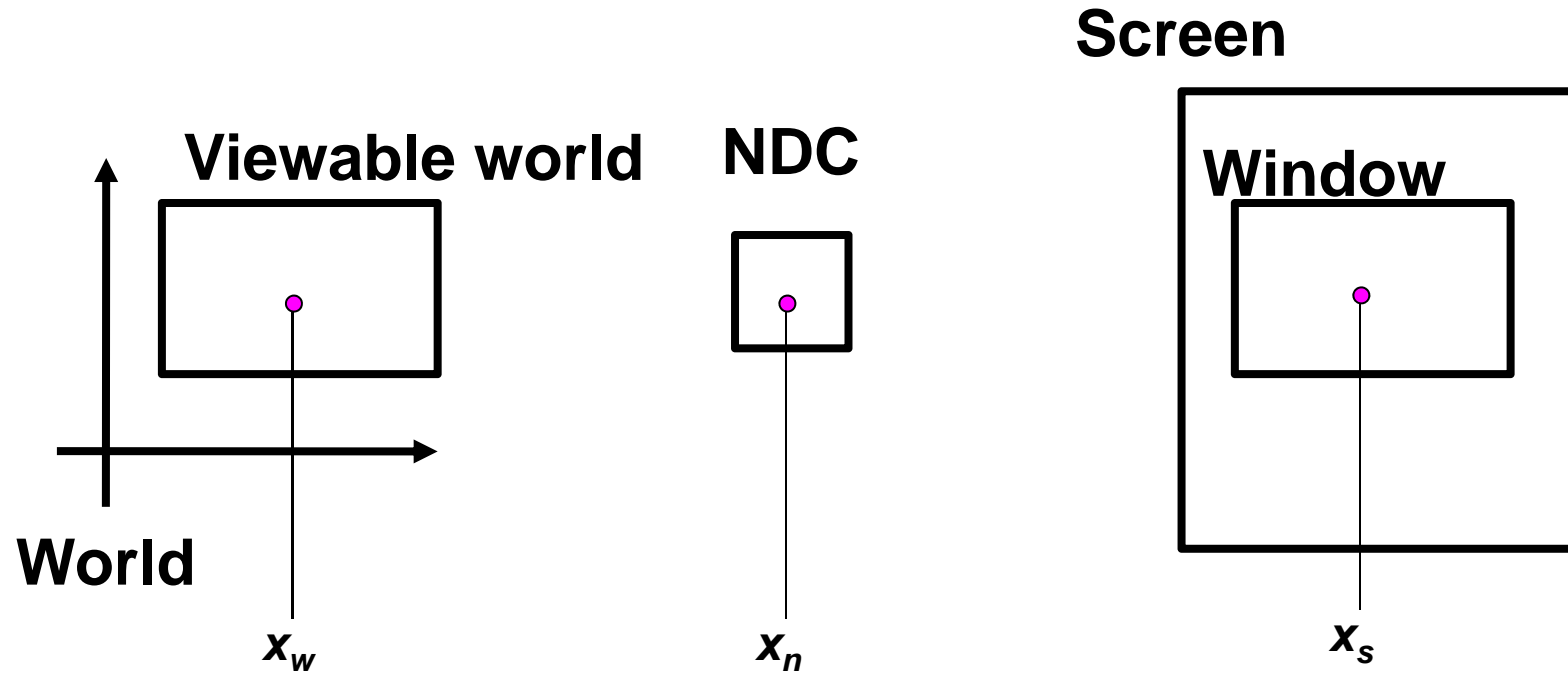
- **Understand the diff. between points and vectors**
- **Understand the frame**
- **Represent transformations in local and global frames**

A Question?

- **Suppose you have 2 frames and you know the coordinates of a point relative in one frame**
 - **How would you compute the coordinate of your point relative to the other frame?**
 - **(Generalized question to the mapping problem that we went over in the class)**

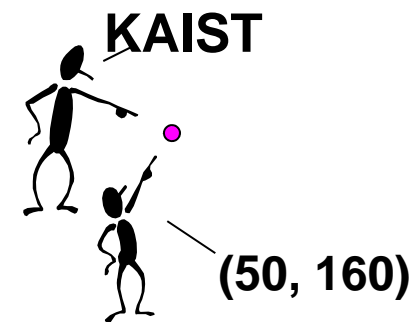


Revisit: Mapping from World to Screen



Geometry

- **A part of mathematics concerned with questions of size, shape, and relative positions of figures**
- **Coordinates are used to represent points and vectors**
 - **We will learn that they are just a naming scheme**
 - **The same point can be described by different coordinates**
 - **Both vectors and points expressed by coordinates, but they are very different**



KAIST

Scalar Fields

- A scalar field **S is a set** on which addition (+) and multiplication (·) are defined and following conditions hold:

- S is closed for addition and multiplication

$$\forall a, b \in S \quad a + b \in S \quad a \cdot b \in S$$

- These operators commute, associate, and distribute

$$\forall a, b, c \in S$$

$$a + b = b + a \quad a \cdot b = b \cdot a$$

$$a + (b + c) = (a + b) + c \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Scalar Fields – cont'd

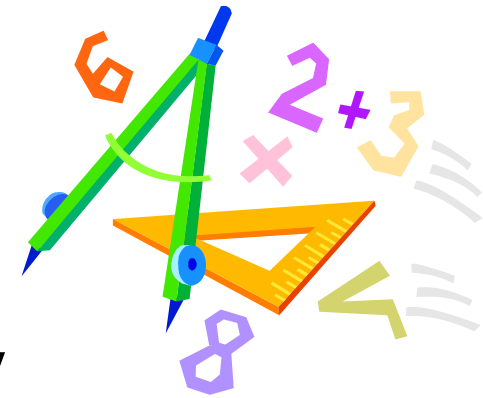
- A scalar field **S is a set** on which addition (+) and multiplication (·) are defined and following conditions hold:
 - Both operators have a unique identity element
$$a + 0 = a, \quad a \cdot 1 = a$$
 - Each element has a unique inverse under both operators

$$a + (-a) = 0, \quad a \cdot a^{-1} = 1$$

Examples of Scalar Fields

- Real numbers
- Complex numbers
(given the standard definitions for addition and multiplication)
- Rational numbers
- Notation: we will represent scalars by lower case letters

a, b, c, ... are scalar variables



Vector Spaces

- **A vector (or linear) space V over a scalar field S consists of a set on which the following two operators are defined and the following conditions hold:**

- **Two operators for vectors:**

- **Vector-vector addition**

$$\forall \vec{u}, \vec{v} \in V \quad \vec{u} + \vec{v} \in V$$

- **Scalar-vector multiplication**

$$\forall \vec{u} \in V, \forall a \in S \quad a\vec{u} \in V$$

- **Notation:**

- **Vector** $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a \quad b \quad c]^t$

Vector Spaces

- **Vector-vector addition**

- **Commutates and associates**

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

- **An additive identity and an additive inverse for each vector**

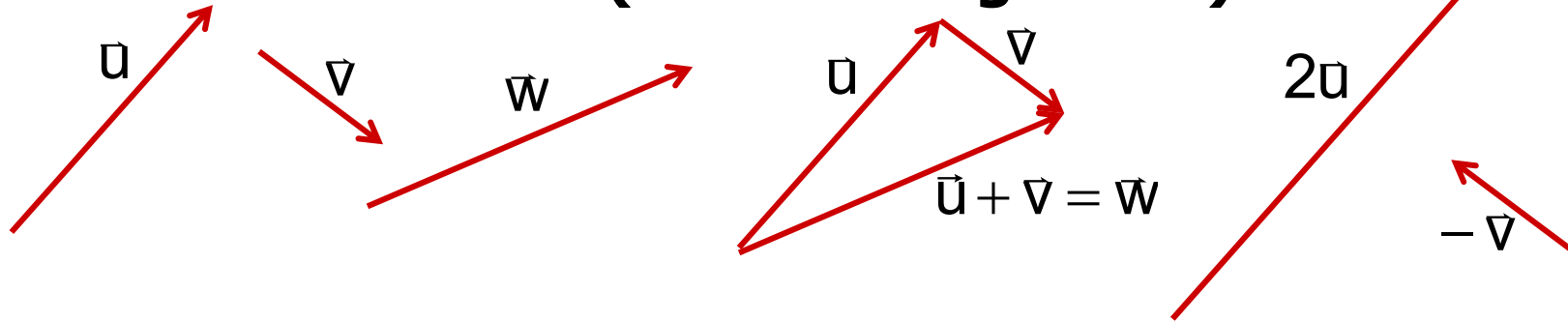
$$\vec{u} + \vec{0} = \vec{u} \quad \vec{u} + (-\vec{u}) = \vec{0}$$

- **Scalar-vector multiplication distributes**

$$(a + b)\vec{u} = a\vec{u} + b\vec{u} \quad a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$

Example Vector Spaces

- **Geometric vectors (directed segments)**



- **N-tuples of scalars**

$$\vec{u} = (1, 3, 7)^t \quad \vec{u} + \vec{v} = (3, 5, 4)^t = \vec{w}$$

$$\vec{v} = (2, 2, -3)^t \quad 2\vec{u} = (2, 6, 14)^t$$

$$\vec{w} = (3, 5, 4)^t \quad -\vec{v} = (-2, -2, 3)^t$$

- **We can use N-tuples to represent vectors**

Basis Vectors

- A **vector basis** is a subset of vectors from V that can be used to generate any other element in V , using just additions and scalar multiplications
- A basis set, $\nabla_1, \nabla_2, \dots, \nabla_n$, is **linearly dependent** if:

$$\exists a_1, a_2, \dots, a_n \neq 0 \quad \text{such that} \quad \sum_{i=1}^n a_i \nabla_i = 0$$

- Otherwise, the basis set is **linearly independent**
 - A linearly independent basis set with i elements is said to *span* an *i-dimensional* vector space

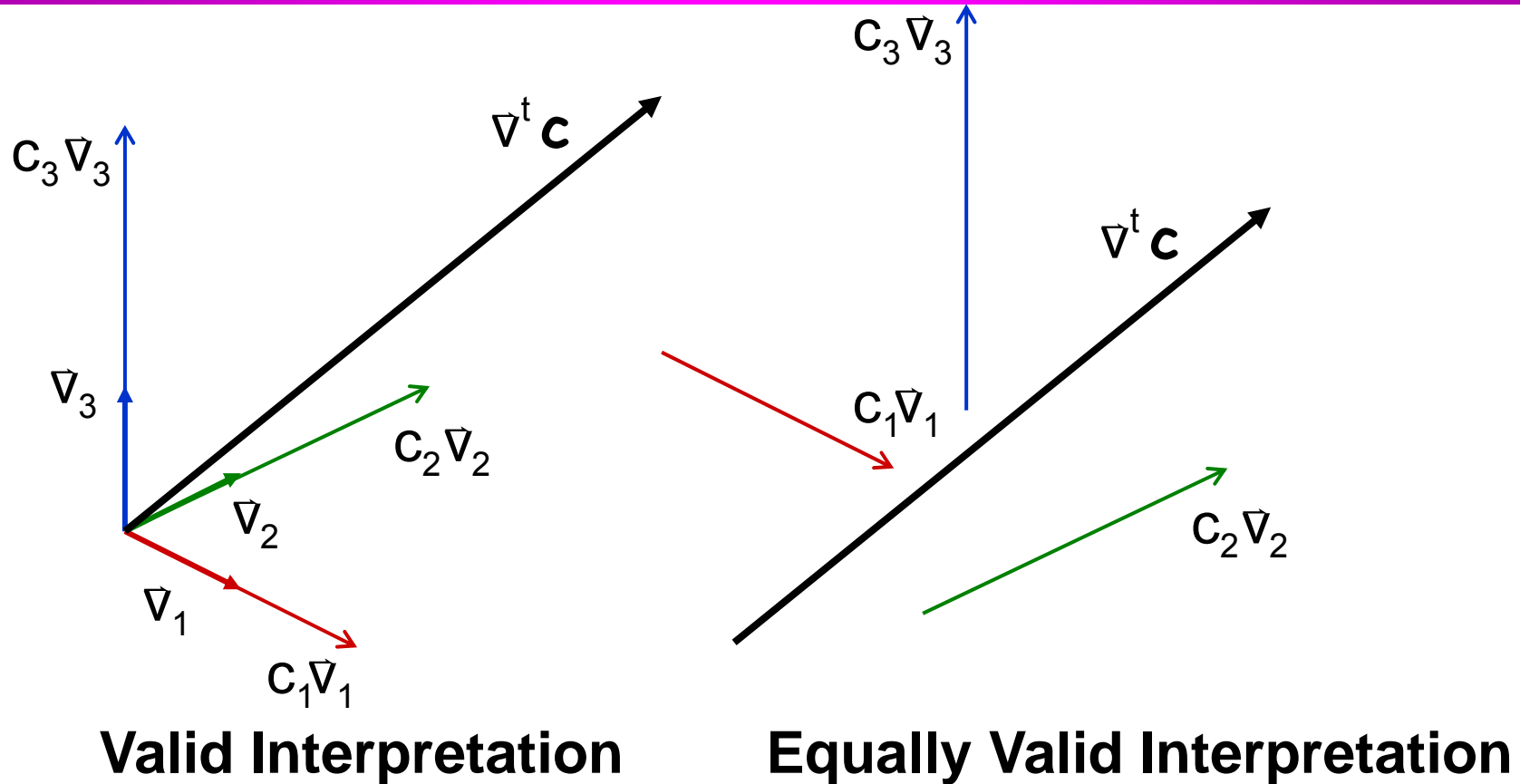
Vector Coordinates

- A linearly independent basis set can be used to uniquely name or address a vector
 - This is done by assigning the vector **coordinates** as follows:

$$\vec{x} = \sum_{i=1}^3 c_i \vec{v}_i = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{v}^t \mathbf{c}$$

- Note: we'll use **bold letters** to indicate tuples of scalars that are interpreted as coordinates
- Our vectors are still abstract entities
 - So how do we interpret the equation above?

Interpreting Vector Coordinates



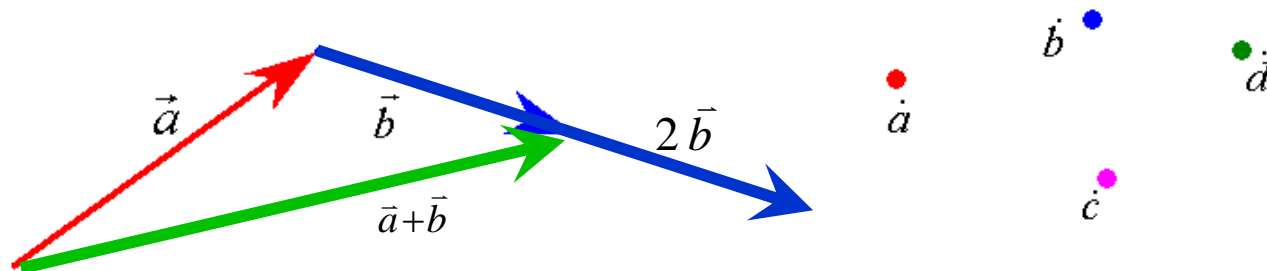
Remember, vectors don't have any notion of position

Points

- **Conceptually, points and vectors are very different**
 - **A point \dot{p} is a place in space**
 - **A vector \vec{v} describes a direction independent of position (pay attentions notations)**

How Vectors and Points Differ

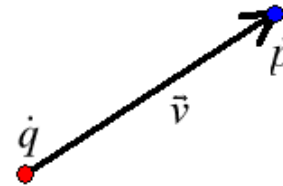
- The operations of addition and multiplication by a scalar are well defined for vectors
 - Addition of 2 vectors expresses the concatenation of 2 “motions”
 - Multiplying a vector by some factor scales the motion
- These operations does not make sense for points



Making Sense of Points

- Some operations **do make sense** for points
 - Compute a vector that describes the motion from one point to another:

$$\vec{p} - \vec{q} = \vec{v}$$



- Find a new point that is some vector away from a given point:

$$\vec{q} + \vec{v} = \vec{p}$$

A Basis for Points

- **Key distinction between vectors and points:**
points are *absolute*, vectors are *relative*
- **Vector space is completely defined by a set of basis vectors**
- **The space that points live in requires the specification of an absolute origin**

$$p = o + \sum_i v_i c_i = \begin{bmatrix} v_1 & v_2 & v_3 & o \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Notice how 4 scalars (one of which is 1) are required to identify a 3D point

Frames

- Points live in *Affine spaces*
- Affine-basis-sets are called *frames*

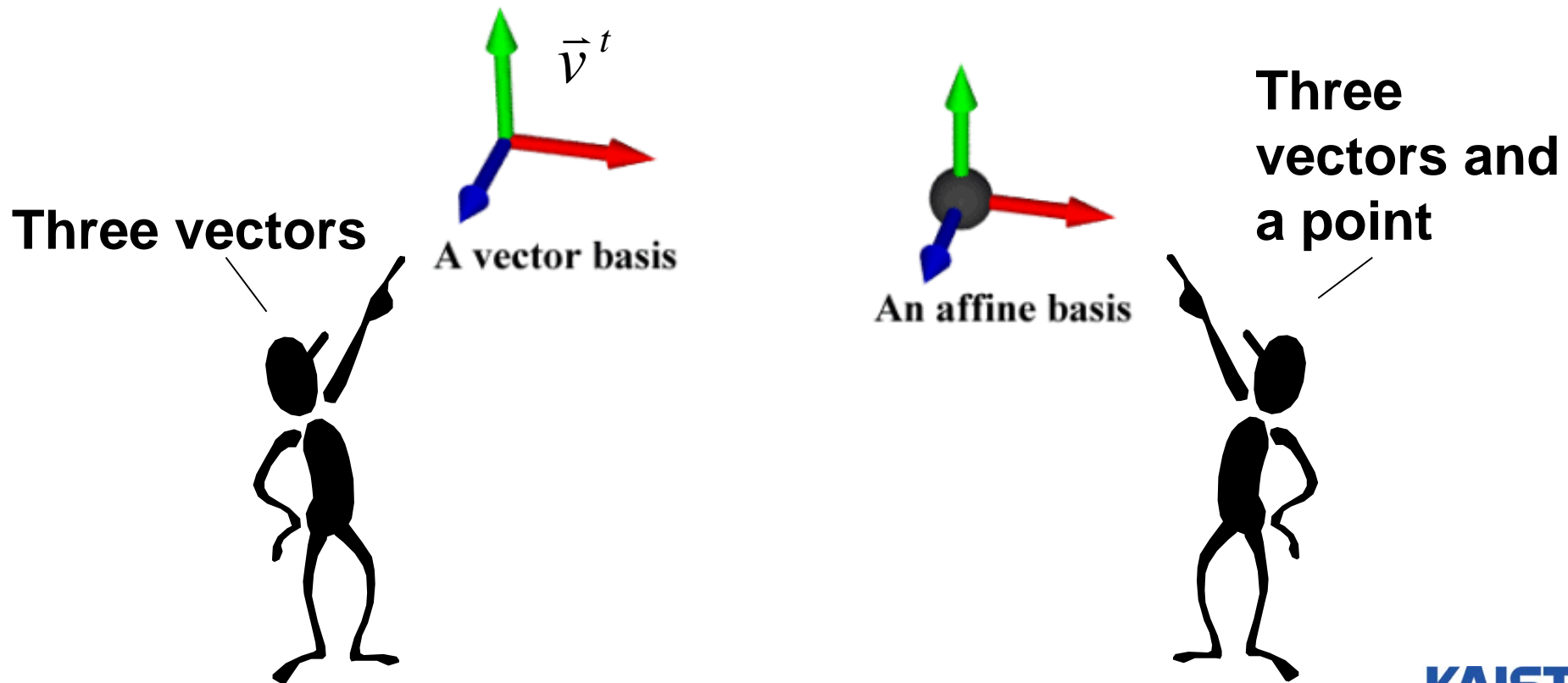
$$\mathbf{f}^t = [\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3 \quad \bar{o}]$$

- Frames can describe vectors as well as points

$$\dot{p} = [\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3 \quad \bar{o}] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \quad \bar{x} = [\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3 \quad \bar{o}] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix}$$

Pictures of Frames

- Graphically, we will distinguish between vector bases and affine bases (frames) using the following convention



A Consistent Model

- **Behavior of affine frame coordinates is completely consistent with our intuition**
 - **Subtracting two points yields a vector**
 - **Adding a vector to a point produces a point**
 - **If you multiply a vector by a scalar you still get a vector**
 - **Scaling points gives a nonsense 4th coordinate element in most cases**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + v_1 \\ a_2 + v_2 \\ a_3 + v_3 \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

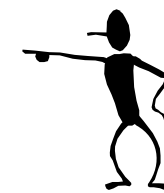
- Notice why we introduce **homogeneous coordinates**, based on simple logical arguments
 - Remember that **coordinates are not geometric**; they are just scales for basis elements
 - Thus, you should not be bothered by the fact that our coordinates suddenly have 4 numbers
- **3D homogeneous coordinates refer to an affine frame with its 3 basis vectors and origin point**
 - 4 coordinates make sense in this aspect
 - 4th coordinate can have one of two values, $[0,1]$, indicating if whether the coordinates name a vector or a point

Affine Combinations

- There are certain situations where it makes sense to scale and add points
 - Suppose you have two points, one scaled by α_1 and the other scaled by α_2
 - If we restrict the sum of these alphas, $\alpha_1 + \alpha_2 = 1$, we can assure that the result will have **1** as it's 4th coordinate value

$$\alpha_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 b_1 \\ \alpha_1 a_2 + \alpha_2 b_2 \\ \alpha_1 a_3 + \alpha_2 b_3 \\ \alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 b_1 \\ \alpha_1 a_2 + \alpha_2 b_2 \\ \alpha_1 a_3 + \alpha_2 b_3 \\ 1 \end{bmatrix}$$

But, is it a point?



Affine Combinations

- Can be thought of as a constrained-scaled addition
 - Defines all points that share the line connecting our two initial points



- Can be extended to 3, 4, or any number of points (e.g., barycentric coordinates)

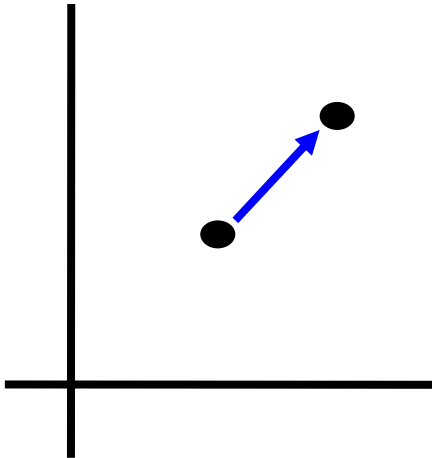
Affine Transformations

- We can apply transformations to points using matrix
 - Need to use 4 by 4 matrices since our basis set has four components
 - Also, limit ourselves to transforms that preserve the integrity of our points and vectors; point to point, vector to vector

$$\dot{p} = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \Rightarrow \dot{p}' = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

- This subset of matrices is called the *affine* subset

An Example



Composing Transformations

- **Represent a series of transformations**
 - E.g., want to translate with T and, then, rotate with R

- **Then, the series is represented by:**

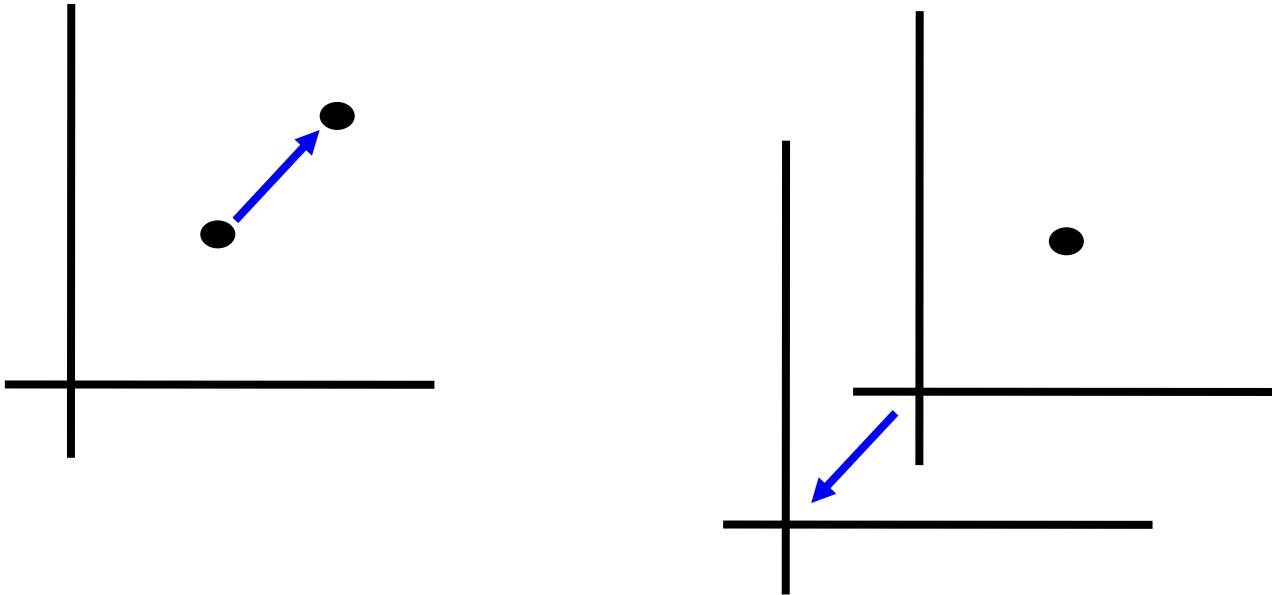
$$\dot{p} = \dot{w}^t c \Rightarrow \dot{p}' = \dot{w}^t R T c = \dot{w}^t (R(Tc)) = \dot{w}^t (Rc') = \dot{w}^t c''$$

- Each step in the process can be considered as a change of coordinates
- **Alternatively, we could have considered the same sequence of operations as:**

$$\dot{p} = \dot{w}^t c \Rightarrow \dot{p}' = \dot{w}^t R T c = ((\dot{w}^t R) T) c = (\dot{m}^t T) c = \dot{e}^t c$$

, where each step is considered as a change of basis

An Example



- **These are alternate interpretations of the same transformations**
 - The left and right sequence are considered as a transformation about a *global frame and local frames*

Same Point in Different Frames

- **Suppose you have 2 frames and you know the coordinates of a point relative in one frame**
 - **How would you compute the coordinate of your point relative to the other frame?**

$$\hat{p} = w^t c = z^t ?$$

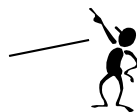
- **Suppose that my two frames are related by the transform S as shown below:**

$$z^t = w^t S \quad \text{and} \quad w^t = z^t S^{-1}$$

- **Then, the coordinate for the point in second frame is simply:**

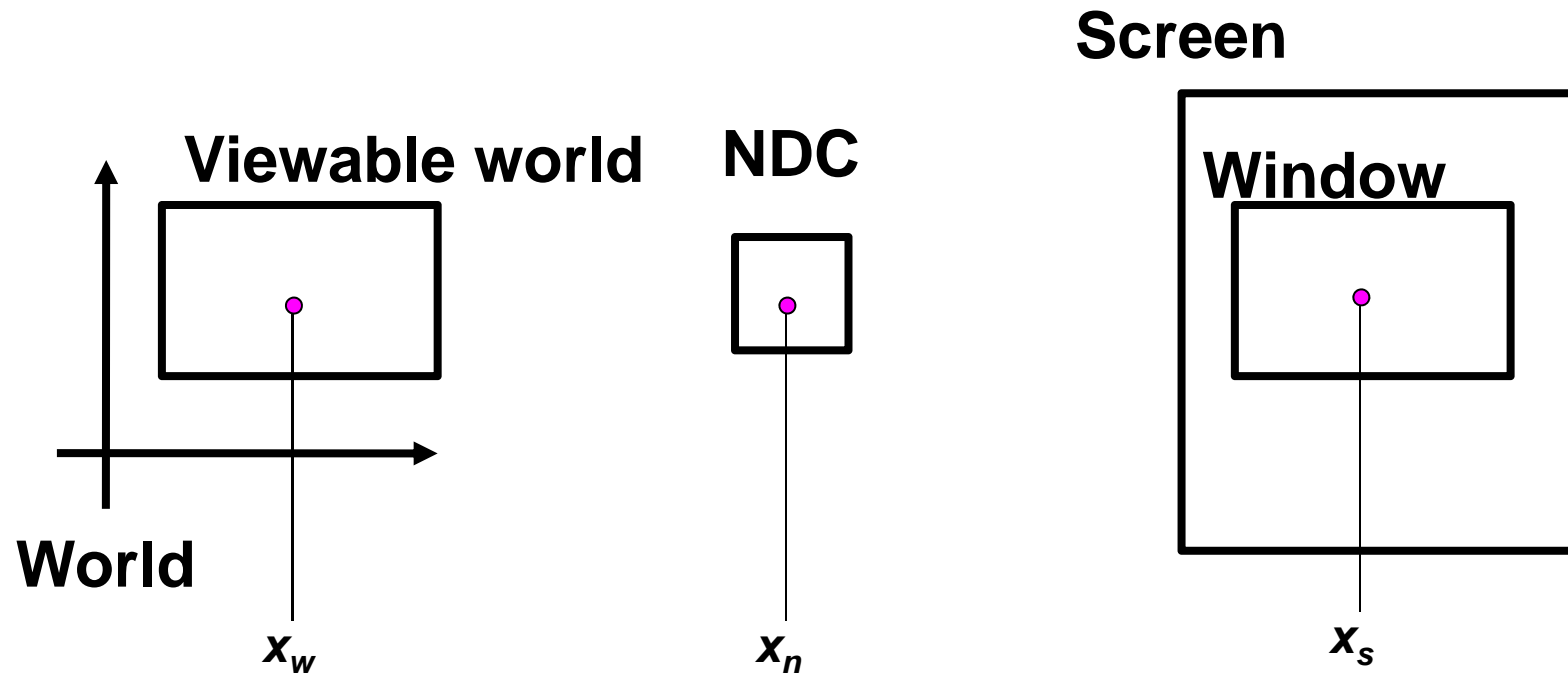
$$\hat{p} = w^t c = z^t S^{-1} c = z^t (S^{-1} c) = z^t d$$

Substitute
for the
frame



Reorganize
&
reinterpret

Revisit: Mapping from World to Screen



Class Objectives were:

- **Understand the diff. between points and vectors**
- **Understand the frame**
- **Represent transformations in local and global frames**

Quiz Assignment



Next Time

- **Modeling and viewing transformations**

Homework

- **Go over the next lecture slides before the class**
- **Watch 2 SIGGRAPH videos and submit your summaries before every Tue. class**

Any Questions?

- **Come up with one question on what we have discussed in the class and submit at the end of the class**
 - **1 for already answered questions**
 - **2 for typical questions**
 - **3 for questions with thoughts or that surprised me**
- **Submit at least four times during the whole semester**